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LETTER TO THE EDITOR

Supergravity coupling to non-linear realisations in two dimensions

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Abstract. The action corresponding to a two-dimensional spinor non-linear realisation of supersymmetry is consistently coupled to supergravity. The resulting formal model has local supersymmetry invariance, which permits gauging away of the fermion.

The non-linear realisations of global supersymmetry (Volkov and Akulov 1973) have recently become of interest in the context of local supersymmetry, i.e. supergravity (Freedman *et al* 1976, Deser and Zumino 1976a), as generic representations of Goldstone fermions when global supersymmetry is broken (Deser and Zumino 1977). Although the general features of the coupling between the fermion and supergravity seem clear, its non-linearity makes it difficult to exhibit explicitly in four dimensions, and indeed even the extension of global invariance to the case of de Sitter space is not straightforward (Zumino 1977). The purpose of this letter is to consider the much simpler, two-dimensional version of this system. While hardly very realistic (there are no Goldstone fermions in two dimensions), this model has the virtue that its coupling to supergravity is obvious and does exhibit, within the usual two-dimensional idiosyncrasies, the formal properties expected in four dimensions. It also provides some hints about the construction of a consistent coupling in the latter case.

The properties and simplifying features of fermions and metrics in two dimensions are discussed in Deser and Zumino (1976b); we need only mention two of them here: of the matrix basis $(1, \gamma_5, \gamma^a)$ only 1 is even, $\bar{\alpha} 1 \beta = \bar{\beta} 1 \alpha$, and the rest are odd for anticommuting Majorana spinors, and the covariant derivative of a spinor in a curved two-space is simply given by $\mathrm{D}_\mu \lambda \equiv (\partial_\mu + \frac{1}{2}\omega_\mu \gamma_5)\lambda$ where ω_μ is the rotation coefficient. The self-coupled fermion action

$$I = -\frac{1}{4a^2} \int \mathrm{d}^2 x \Sigma^{\mu\nu} \Sigma_{ab} (\delta_\mu{}^a - \mathrm{i}a^2 \bar{\lambda} \gamma^a \partial_\mu \lambda)(\delta_\mu{}^b - \mathrm{i}a^2 \bar{\lambda} \gamma^b \partial_\mu \lambda) \tag{1}$$

is our starting point. It is invariant under the non-linear transformation

$$\delta\lambda = \alpha - \mathrm{i}a^2 (\bar{\alpha}\gamma^\mu \lambda)\partial_\mu \lambda \tag{2}$$

where α is a constant spinor, which generalises the trivial invariance under $\delta\lambda = \alpha$ for the free Dirac action. The invariance is obvious when I is written in terms of forms in

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superspace (Volkov and Akulov 1973). The action (1) may be thought of as describing a single two-component Majorana spinor λ, which has one degree of freedom† *a priori.*

Since the non-linear part of (2) is really just a coordinate translation with a field dependent $\delta x^\mu \sim (\bar\alpha\gamma^\mu\lambda)$, it suggests that we may linearise the transformation (2) by introducing a *vierbein* field to perform the coordinate change for us, i.e. write (1) in the generally covariant form

$$I=-\frac{1}{4a^2}\int d^2x\,\Sigma^{\mu\nu}\Sigma_{ab}(e_\mu{}^a-ia^2\bar\lambda\gamma^a\partial_\mu\lambda)(e_\nu{}^b-ia^2\bar\lambda\gamma^b\partial_\nu\lambda) \tag{3}$$

in terms of the *vierbein* $e_\mu{}^a$; note that no covariant derivative appears in I since $\bar\lambda\gamma^a\gamma^5\lambda=0$ for a Majorana spinor. Then if we let

$$\delta e_\mu{}^a=ia^2\bar\alpha\gamma^a\partial_\mu\lambda \qquad \delta\lambda=\alpha \tag{4}$$

for α constant, (3) is still invariant, though constant α and use of $\partial_\mu\lambda$ rather than $D_\mu\lambda$ in $\delta e_\mu{}^a$ is only meaningful at flat space. (In particular $D_\mu\alpha=0$ implies $[D_\mu, D_\nu]\alpha = R_{\mu\nu ab}\epsilon^{ab}\gamma_5\alpha=0$.) This step does suggest that we extend the coupling to include supergravity and thereby achieve local supersymmetry under arbitrary $\alpha(x)$ by introducing the spin-$\frac{3}{2}$ gauge field $\psi_\mu(x)$ in the combination $\partial_\mu\lambda\to(\partial_\mu\lambda-\psi_\mu)$.

Our final action then is given by

$$I=-\frac{1}{4a^2}\int d^2x\,\Sigma^{\mu\nu}\Sigma_{ab}[e_\mu{}^a-ia^2\bar\lambda\gamma^a(\partial_\mu\lambda-\psi_\mu)][e_\nu{}^b-ia^2\bar\lambda\gamma^b(\partial_\nu\lambda-\psi_\nu)] \tag{5}$$

and there is no pure supergravity part of the action since we are in two dimensions‡. The action (5) actually possesses two slightly different local invariances under arbitrary $\alpha(x)$. Their consequences are the same, but they may be different in four dimensions. The first is the most obvious,

$$\begin{aligned}&\delta e_\mu{}^a=ia^2\bar\alpha\gamma^a D_\mu\lambda-ia^2\bar\alpha\gamma^a\psi_\mu\\ &\delta\lambda=\alpha(x) \qquad \delta\psi_\mu=D_\mu\alpha;\end{aligned} \tag{6}$$

the second keeps the λ-dependent part of $\delta e_\mu{}^a$ of the pure coordinate transformation type,

$$\begin{aligned}&\delta e_\mu{}^a=ia^2D_\mu(\bar\alpha\gamma^a\lambda)-ia^2\bar\alpha\gamma^a\psi_\mu\\ &\delta\lambda=\alpha(x) \qquad \delta\psi_\mu=2D_\mu\alpha.\end{aligned} \tag{7}$$

Here we need not specify whether the connection ω_μ in D_μ contains torsion. We emphasise that this local invariance holds for arbitrary values of the symmetry breaking parameter a, which has been assumed to be the case in four dimensions. The invariance under (6) is obvious since

$$\begin{aligned}&\delta[e_\mu{}^a-ia^2\bar\lambda\gamma^a(\partial_\mu\lambda-\psi_\mu)]\\ &\qquad=ia^2\bar\alpha\gamma^a(D_\mu-\partial_\mu)\lambda-ia^2\bar\lambda\gamma^a(\partial_\mu-D_\mu)\alpha\\ &\qquad=\tfrac{1}{2}ia^2\omega_\mu(\bar\alpha\gamma^a\gamma_5\lambda+\bar\lambda\gamma^a\gamma_5\alpha)\\ &\qquad\equiv 0\end{aligned} \tag{8}$$

† We are thus not dealing here with a string model in which λ is also a Minkowski vector in the embedding space.

‡ We have not pursued 'cosmological' supergravity in two dimensions, because its Lagrangian $\Lambda e+\frac{1}{2}im\epsilon^{\mu\nu}\bar\psi_\mu\gamma_5\psi_\nu$ is invariant under rather unnatural transformations, i.e. either ($\delta e_\mu{}^a=ib\alpha\gamma^a\psi_\mu$, $\delta\psi_\mu=c\gamma_\mu\alpha$) with $\Lambda b-mc=0$ and no $D_\mu\alpha$ part of $\delta\psi_\mu$ or a $\delta e_\mu{}^a$ depending on derivatives of ψ_μ.

while the invariance under (7) is checked by noting that $\bar{\alpha}\gamma^a\lambda$ is a world scalar-local vector, so that $D_\mu(\bar{\alpha}\gamma^a\lambda) = \partial_\mu(\bar{\alpha}\gamma^a\lambda) + \omega_\mu(\bar{\alpha}\gamma^a\gamma_5\lambda)$. Thus, in this case,

$$\begin{aligned}\delta[e_\mu{}^a - \mathrm{i}a^2\bar{\lambda}\gamma^a(\partial_\mu\lambda - \psi_\mu)] \\ &= \mathrm{i}a^2\partial_\mu(\bar{\alpha}\gamma^a\lambda) + \mathrm{i}a^2\omega_\mu(\bar{\alpha}\gamma^a\gamma_5\lambda) - \mathrm{i}a^2\bar{\alpha}\gamma^a\partial_\mu\lambda - \mathrm{i}a^2\bar{\lambda}\gamma^a(\partial_\mu - 2\partial_\mu - \omega_\mu\gamma_5)\alpha \\ &= 0. \end{aligned} \tag{9}$$

We have thus achieved coupling to supergravity. At this point there arises a very surprising feature. The λ degree of freedom can be removed entirely, as expected of a Goldstone particle, by simply using $\alpha(x)$ to set $\lambda = 0$, reducing the action to the trivial form $I = \frac{1}{2}a^2\int d^2xe$, which implies $e_\mu{}^a = 0$. But this means that the λ degree of freedom has not re-appeared in the Higgs-absorbing gauge fields, simply because in two dimensions these are not dynamical. There is a discontinuity between the original global action (1), and its supergravity-coupled form (5) as far as degrees of freedom are concerned. Nevertheless the Higgs mechanism does work to remove the fermion into the 'accidentally' empty gauge fields. Compare this with the following analogue. In any dimensions, a free massless scalar field, $L = -\frac{1}{2}(\partial_\mu\phi)^2$ is invariant under $\delta\phi(x) =$ constant. Now 'gauge' it to be $L = -\frac{1}{2}(\partial_\mu\phi - V_\mu)^2$, which is clearly invariant under $\delta\phi = \sigma(x)$, $\delta V_\mu = \partial_\mu\sigma$. But in the gauge $\phi = 0$, the field equations $V_\mu = \partial_\mu\phi$ imply that V_μ also vanishes since it has no kinetic term and there is a discontinuity between the free and the gauged systems. Here, of course, we could (and should) add a kinetic term for V_μ even in two dimensions. If $-\frac{1}{4}(\partial_\mu V_\nu - \partial_\nu V_\mu)^2$ is added, then in $\phi = 0$ gauge, one now obtains a massive vector field, which has precisely one (longitudinal) degree of freedom, and the count of degrees of freedom is continuous between the free and coupled models. In our case, no kinetic terms exist, and the coupled action (5) is purely formal†.

Finally, we make some remarks about the four-dimensional case. Here, the total Lagrangian will be the sum of the usual supergravity part, L_{sg}, and the non-linear matter action. The *ansatz* equivalent to (5),

$$L_M = -\frac{1}{2a^2}\frac{1}{4!}\Sigma^{\mu\nu\kappa\lambda}\Sigma_{abcd}[e - \mathrm{i}a^2\bar{\lambda}\gamma(D\lambda - \psi)]^4, \tag{10}$$

with the transformation rules (6) or (7) almost works. The only problems are that we must take into account variations of ω in the explicit ω-dependent terms in (10), through $\bar{\lambda}\gamma^a\omega_{\mu cd}\sigma^{cd}\lambda = \bar{\lambda}\gamma_5\gamma_b\lambda\ \epsilon^{abcd}\omega_{\mu cd}$; this leads to terms like $\bar{\lambda}\lambda\bar{\alpha}D\psi$ times the minor of the determinant. Conversely the new, λ-dependent part of $\delta e_\mu{}^a$ will get contributions from L_{sg}. Note that in the form (7), we have precisely the correct $(\delta e, \delta\psi)$ transformation rules together with a pure coordinate change, $\delta_c e_\mu{}^a$. The latter leaves the Einstein action in second-order form identically invariant, by the contracted Bianchi identity, but gives contributions of the type $\bar{\psi}D\psi\ D\lambda$ from the spin-$\frac{3}{2}$ action. It will therefore probably be necessary to include complicated contact terms. However, the form (10) certainly embodies the beginnings of the correct action.

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† It is possible that the field equations following from (1) also only have trivial solutions, but we have not been able to show this. Certainly (1) is not equivalent to (5) in any ψ_μ or $e_\mu{}^a$ gauge, since neither field can be gauged away from (5).

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Appendix

We sketch here the direct verification that (1) is invariant under (2) and also exhibit the Noether current associated with the global supersymmetry (2) and verify its conservation property.

It is convenient to rewrite the quartic part of the action (1) in either one of the following forms:

$$I_1 = -\tfrac{1}{4}a^2 \int d^2x\bar{\lambda}\lambda \,.\, \Sigma^{\mu\nu}\overline{\partial_\mu\lambda}\gamma_5\partial_\nu\lambda \tag{A.1a}$$

$$I_1 = -\tfrac{1}{4}a^2 \int d^2x(\bar{\lambda}\gamma^\mu\partial_\mu\lambda \,.\, \bar{\lambda}\gamma^\nu\partial_\nu\lambda - \bar{\lambda}\gamma^\mu\partial_\nu\lambda \,.\, \bar{\lambda}\gamma^\nu\partial_\mu\lambda). \tag{A.1b}$$

To obtain these from (1) we made some Fierz re-arrangements and used a little Dirac algebra. Separating $\delta\lambda$ into its linear and cubic parts as $\delta_0\lambda = \alpha$ and $\delta_1\lambda = -ia^2(\bar{\alpha}\gamma^\mu\lambda)\partial_\mu\lambda$, we immediately check $\delta_0 I_0 = 0$ and from (A.1*b*)

$$\delta_0 I_1 = -\tfrac{1}{2}a^2 \int d^2x(\bar{\alpha}\gamma^\mu\partial_\mu\lambda \,.\, \bar{\lambda}\gamma^\nu\partial_\nu\lambda - \bar{\alpha}\gamma^\mu\partial_\nu\lambda \,.\, \bar{\lambda}\gamma^\nu\partial_\mu\lambda). \tag{A.2}$$

The last line, by inspection, is precisely the negative of $\delta_1 I_0$. The quintic part, $\delta_1 I_1$, of the variation is the one that gives the most trouble. But many terms drop simply by making use of the fact that cubic or higher powers of a spinor vanish in two dimensions. Using (A.1*a*) for I_1, the remaining terms, after some re-arrangement of indices, become

$$\delta_1 I_1 = -\tfrac{1}{4}ia^4 \int d^2x\bar{\lambda}\lambda\,\Sigma^{\mu\nu}\bar{\alpha}\gamma^\rho\psi_{[\mu\nu\rho]} \tag{A.3}$$

where

$$\psi_{[\mu\nu\rho]} = \partial_\mu\lambda \,.\, \overline{\partial_\nu\lambda}\gamma_5\partial_\rho\lambda + \partial_\nu\lambda \,.\, \overline{\partial_\rho\lambda}\gamma_5\partial_\mu\lambda + \partial_\rho\lambda \,.\, \overline{\partial_\mu\lambda}\gamma_5\partial_\nu\lambda \tag{A.4}$$

is totally antisymmetric in three indices, and thus vanishes in two dimensions.

It is puzzling at first sight how a conserved current associated with the invariance (2) can exist, because the field equation reads

$$i\gamma^\mu\partial_\mu\lambda - \tfrac{1}{2}a^2\partial_\mu(\bar{\lambda}\lambda)\Sigma^{\mu\nu}\gamma_5\partial_\nu\lambda + \tfrac{1}{2}a^2\lambda \,.\, \Sigma^{\mu\nu}\overline{\partial_\mu\lambda}\gamma_5\partial_\nu\lambda = 0. \tag{A.5}$$

While the first and second terms are already divergences of a vector, the third term is not, and it is only by use of (A.5) itself that it may be shown to be a divergence (on shell). More systematically we may conveniently read off the Noether current from the variation of the action (5) with respect to ψ_μ, evaluated at $\psi_\mu = 0$. It has the form:

$$J^\mu = \gamma^\mu\lambda + ia^2\bar{\lambda}\lambda \,.\, \Sigma^{\mu\nu}\gamma_5\partial_\nu\lambda. \tag{A.6}$$

Its divergence then reads

$$\partial_\mu J^\mu = \gamma^\mu\partial_\mu\lambda + ia^2\partial_\mu(\bar{\lambda}\lambda)\Sigma^{\mu\nu}\gamma_5\partial_\nu\lambda. \tag{A.7}$$

Upon inserting the field equation (A.5), this becomes

$$\partial_\mu J^\mu = \tfrac{1}{2}ia^2[\partial_\mu(\bar{\lambda}\lambda)\Sigma^{\mu\nu}\gamma_5\partial_\mu\lambda + \lambda \,.\, \Sigma^{\mu\nu}\overline{\partial_\mu\lambda}\gamma_5\partial_\nu\lambda]. \tag{A.8}$$

The first term on the right-hand side, when Fierz transformed, just cancels the second term leaving behind a contribution of the form

$$\partial_\mu J^\mu = -\tfrac{1}{2}\mathrm{i}a^2\gamma^\rho\lambda\,.\,\Sigma^{\mu\nu}\overline{\partial_\mu\lambda}\gamma_\rho\gamma_5\partial_\nu\lambda. \tag{A.9}$$

One way to see the vanishing of this quantity is to use the fact that polynomials of the form $\lambda\,(\overline{\not\partial\lambda})(\not\partial\lambda)$ vanishes on shell because each $\not\partial\lambda \equiv \gamma^\mu\partial_\mu\lambda$ is proportional, by (A.5), to one undifferentiated λ, and so the total is cubic in λ. Using the identity $\epsilon^{\mu\nu}\gamma_5 = \gamma^\mu\gamma^\nu - \eta^{\mu\nu}$, (A.9) reduces to

$$\partial_\mu J^\mu = -\mathrm{i}a^2\gamma^\mu\lambda\,.\,\overline{\partial_\mu\lambda}\gamma^\nu\partial_\nu\lambda. \tag{A.10}$$

But on shell $\overline{\partial_\rho\lambda}\gamma^\nu\partial_\nu\lambda$ is proportional, after some re-arrangements, to a totally antisymmetric three-index quantity, namely to

$$\tfrac{1}{2}\mathrm{i}a^2\Sigma^{\mu\nu}\bar{\lambda}\psi_{[\mu\nu\rho]}$$

with $\psi_{[\mu\nu\rho]}$ as defined by (A.4), and so must be identically zero.

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